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κ-deformation of Poincaré superalgebra with classical Lorentz subalgebra and its graded bicrossproduct structure

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Abstract. The κ-deformed $D = 4$ Poincaré superalgebra written in Hopf superalgebra form is transformed to the basis with classical Lorentz subalgebra generators. We show that in such a basis the κ-deformed $D = 4$ Poincaré superalgebra can be written as a graded bicrossproduct. We show that the κ-deformed $D = 4$ superalgebra acts covariantly on a κ-deformed chiral superspace.

1. Introduction

The quantum deformations of Lie algebras and Lie groups [1–4] have recently been applied to the $D = 4$ relativistic symmetries and its supersymmetric extensions [5–20]. Two types of quantum deformations occur in these considerations:

(i) q-deformations, with dimensionless parameter q [5–13].
In this framework the 4-momenta do not commute, and usually the introductions of Hopf algebra structure implies the appearance of additional generators (for $D = 4$ Poincaré algebra the dilatation generators [5, 6] and for $D = 4$ Poincaré superalgebra the dilatations and chirality generators [6, 10, 13].

(ii) κ-deformations, with the deformation parameter κ describing the fundamental mass parameter [14–20].

Following the formulation of $D = 4$ κ-deformed Poincaré algebra [14–16] recently also the κ-deformation of $D = 4$, $N = 1$ Poincaré superalgebra was also given [17]. Both deformations were firstly obtained in the framework of Hopf (super)algebras by the quantum contraction procedure of $\mathcal{U}_q(O(3, 2))$ and $\mathcal{U}_q(OSp(1|4))$ and have the following properties:

(i) The 4-momenta remain commutative, but non-cocommutative.
(ii) The three-dimensional rotations remain classical as Hopf algebras.
(iii) The Lorentz generators do not form a subalgebra (neither the Lie subalgebra nor the Hopf subalgebra).

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The first two properties imply that the deformation is 'mild', and does not affect the rotational symmetry of non-relativistic physics. Property (iii) is not convenient from a physical point of view—in particular, there are difficulties with the interpretation of finite κ-deformed Lorentz transformations which do not form a Lie group [18]. Recently, however, work by Majid and Ruegg [19] has given the basis of quantum κ-Poincaré algebra, describing it as a bicrossproduct of the classical Lorentz Hopf algebra $O(3,1)$ with the Hopf algebra T_4^κ of commuting 4-momenta equipped with κ-deformed coproduct

$$\mathcal{P}_4^\kappa = O(1,3) \triangleright\!\!\blacktriangleleft T_4^\kappa\,. \tag{1.1}$$

In such a framework the classical Lorentz algebra $O(1,3)$ (but not a classical Hopf algebra $O(1,3)$!) is the subalgebra of $\mathcal{P}_4^\kappa$, and T_4^κ forms a Hopf subalgebra of $\mathcal{P}_4^\kappa$.

The aim of this paper is to find an analogous basis for a κ-deformed Poincaré superalgebra, with a classical Lorentz subalgebra and commuting 4-momenta, which supersymmetrize the Majid–Ruegg bicrossproduct basis for κ-Poincaré algebra†. Such a formulation is derived (see section 2) by nonlinear change of the basis of κ-Poincaré superalgebra, obtained previously in [17] from the quantum contraction of $\mathcal{U}_q(OSp(1;4))$. It appears that the κ-Poincaré superalgebra $\mathcal{P}_{4;1}^\kappa$ can be written, for example, as the following graded bicrossproduct‡ which extends supersymmetrically the formula (1.1):

$$\mathcal{P}_{4;1}^\kappa = O(1,3;2) \triangleright\!\!\blacktriangleleft T_{4;2}^\kappa \tag{1.2}$$

where $O(1,3;2)$ is the classical superextension of the Lorentz algebra: $(\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1))$§:

$$\left[M_{\mu\nu}^{(0)}, M_{\rho\tau}^{(0)}\right] = \mathrm{i}\left(\eta_{\mu\tau}M_{\nu\rho}^{(0)} + \eta_{\nu\rho}M_{\mu\tau}^{(0)} - \eta_{\mu\rho}M_{\nu\tau}^{(0)} - \eta_{\nu\tau}M_{\mu\rho}^{(0)}\right) \tag{1.3}$$

with two complex supercharges Q_α ($\alpha = 1,2$) satisfying the relations

$$\left[M_{\mu\nu}^{(0)}, Q_\alpha^{(0)}\right] = \tfrac{1}{2}\mathrm{i}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^{(0)} \qquad \left\{Q_\alpha^{(0)}, Q_\beta^{(0)}\right\} = 0 \tag{1.4a}$$

and $\overline{T}_{4;2}^k$ describes the complex Hopf superalgebra ($\mu,\nu = 0,1,2,3$)

$$\left\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0 \qquad \left[\overline{Q}_{\dot\alpha}, P_\mu\right] = [P_\mu, P_\nu] = 0 \tag{1.4b}$$

supplemented by the following coproducts

$$\begin{aligned}
\Delta P_i &= \mathrm{e}^{-P_0/\kappa} \otimes P_i + P_i \otimes \mathbf{1} \\
\Delta P_0 &= P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0 \\
\Delta \overline{Q}_{\dot\alpha} &= \overline{Q}_{\dot\alpha} \otimes \mathrm{e}^{-P_0/2\kappa} + \overline{Q}_{\dot\alpha} \otimes \mathbf{1}\,.
\end{aligned} \tag{1.5}$$

In section 3 we shall show that the κ-deformation of the $N = 1$ Poincaré superalgebra can be described by the κ-dependent action of the superalgebra $O(1,3;2)$ on $\overline{T}_{4;2}^k$, modifying in the algebraic sector the classical $O(1,3;2)$-covariance relations for the $\overline{T}_{4;2}^k$ generators, as well as κ-dependent coaction of $\overline{T}_{4;2}^k$ on $O(1,3;2)$, modifying the classical coproducts of the $O(1,3;2)$ generators.

The bicrossproduct structure of $\mathcal{P}_{4;1}^\kappa$ implies that the dual Hopf algebra $(\mathcal{P}_{4;1}^\kappa)^*$ describing the quantum $N = 1$ Poincaré supergroup also has a bicrossproduct structure (see e.g. [21])

$$\left(\mathcal{P}_{4;1}^\kappa\right)^* = (O(1,3;2))^* \blacktriangleright\!\!\triangleleft \left(\overline{T}_{4;2}^k\right)^* \tag{1.6}$$

† By supersymmetrization of κ-Poincaré algebra $\mathcal{P}_4^\kappa$ we mean the κ-Poincaré superalgebra $\mathcal{P}_{4;1}^\kappa$ which after formally putting in the bosonic sector the supercharges equal to zero reduces to $\mathcal{P}_4^\kappa$.

‡ Here we give only one possibility—in fact there are four ways of expressing $\mathcal{P}_{4;1}^\kappa$ as a graded bicrossproduct (see section 3).

§ We denote by $I_A^{(0)}$ the generators of classical Lie Hopf (super)algebras, with primitive coproducts $\Delta(I_A^{(0)}) = 1 \otimes I_A^{(0)} + I_A^{(0)} \otimes 1$.

where $(\overline{T}^{k}_{4;2})^*$ describes a κ-deformed complex chiral superspace $(x_\mu, \overline{\theta}_{\dot\alpha})$ on which the κ-deformed superalgebra $\mathcal{P}^{\kappa}_{4;1}$ acts covariantly. The κ-deformed superspace has been introduced recently in [20], but only the bicrossproduct structure of $\mathcal{P}^{\kappa}_{4;1}$ permits us to show that its chiral part transforms covariantly under the κ-deformed supersymmetry transformations. These transformations, obtained by the relation (1.2) are described in section 4. In such a way we have all ingredients which are needed for the construction of a κ-deformed chiral superfields formalism, which we shall present in our next publications.

2. The κ-deformed $N = 1$, $D = 4$ Poincaré superalgebra with classical Lorentz generators

Our starting point are the formulae describing a κ-deformed $N = 1$ $D = 4$ Poincaré superalgebra [17] with Lorentz generators $M_{\mu\nu} = (M_i, L_i)$, the 4-momentum sector $P_\mu = (P_i, P_0)$ and the fermionic supercharges $(Q_\alpha, \overline{Q}_{\dot\alpha})$ $(\alpha = 1, 2;\ Q^+_\alpha = \overline{Q}_{\dot\alpha})$. We recall that the κ-deformation takes the form of a non-cocommutative Hopf superalgebra, with the $O(3)$ sector (described by the generators M_i) classical in algebra and co-algebra sectors.

Let us introduce the following two complex Lorentz boosts:

$$L_i^{(\pm)} = L_i \pm \frac{\mathrm{i}}{8\kappa} T_i \tag{2.1}$$

where

$$T_i = Q^\alpha (\sigma_\mu)_{\alpha\dot\beta} \overline{Q}^{\dot\beta} \tag{2.2}$$

which are complex-conjugated to each other $((L_i^{(+)})^+ = L_i^{(-)})$. Instead of the relations (see [17])

$$[L_i, L_j] = -\mathrm{i}\epsilon_{ijk}\left(M_k \cosh\frac{P_0}{\kappa} - \frac{1}{8\kappa}T_k \sinh\frac{P_0}{2\kappa} + \frac{1}{16\kappa^2}P_k(T_0 - 4\vec{P}\vec{M})\right) \tag{2.3a}$$

$$\begin{aligned}\Delta(L_i) = {} & L_i \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes L_i + \frac{1}{2\kappa}\epsilon_{ijk}(P_j \otimes M_k\, \mathrm{e}^{P_0/2\kappa} + M_j\, \mathrm{e}^{-P_0/2\kappa} \otimes P_k) \\ & + \frac{\mathrm{i}}{8\kappa}(\sigma_i)_{\dot\alpha\beta}\left(\overline{Q}_{\dot\alpha}\, \mathrm{e}^{-P_0/4\kappa} \otimes Q_\beta\, \mathrm{e}^{P_0/4\kappa} + Q_\beta\, \mathrm{e}^{-P_0/4\kappa} \otimes \overline{Q}_{\dot\alpha}\, \mathrm{e}^{P_0/4\kappa}\right)\end{aligned} \tag{2.3b}$$

one gets

$$\left[L_i^{(\pm)}, L_j^{(\pm)}\right] = -\mathrm{i}\epsilon_{ijk}\left(M_k \cosh\frac{P_0}{\kappa} - \frac{1}{4\kappa^2}P_k(\vec{P}\vec{M})\right) \tag{2.4}$$

$$\begin{aligned}\Delta L_i^{(+)} = {} & L_i^{(+)} \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes L_i^{(+)} + \frac{1}{2\kappa}\epsilon_{ijk}\left(P_j \otimes M_k\, \mathrm{e}^{P_0/2\kappa} + M_j\, \mathrm{e}^{-P_0/2\kappa} \otimes P_k\right) \\ & + \frac{\mathrm{i}}{4\kappa}(\sigma_i)_{\dot\alpha\beta}\mathrm{e}^{-P_0/4\kappa}\overline{Q}_{\dot\alpha} \otimes \mathrm{e}^{P_0/4\kappa}Q_\beta\end{aligned} \tag{2.5}$$

$$\begin{aligned}\Delta L_i^{(-)} = {} & L_i^{(-)} \otimes \mathrm{e}^{P_0/2\kappa} + \mathrm{e}^{-P_0/2\kappa} \otimes L_i^{(-)} + \frac{1}{2\kappa}\epsilon_{ijk}\left(P_j \otimes M_k \mathrm{e}^{P_0/2\kappa} + M_j\, \mathrm{e}^{-P_0/2\kappa} \otimes P_k\right) \\ & - \frac{\mathrm{i}}{4\kappa}(\sigma_i)_{\dot\alpha\beta}\mathrm{e}^{-P_0/4\kappa}Q_\beta \otimes \mathrm{e}^{P_0/4\kappa}\overline{Q}_{\dot\alpha}\,.\end{aligned} \tag{2.6}$$

Because the supercharges commute with the 4-momenta one obtains the following unchanged relations:

$$\left[L_i^{(\pm)}, P_j\right] = \mathrm{i}\kappa\delta_{ij}\sinh\frac{P_0}{\kappa} \qquad \left[L_i^{(\pm)}, P_0\right] = \mathrm{i}P_i\,. \tag{2.7}$$

Further one can calculate the formulae in the supercharge sector. The modification (2.1) of the boost operators leads to the following covariance relations:

$$\left[L_i^{(\pm)}, Q_\alpha\right] = -\frac{\mathrm{i}}{2}\mathrm{e}^{\pm P_0/2\kappa}(\sigma_i Q)_\alpha \pm \frac{\mathrm{i}}{4\kappa}P_i Q_\alpha \pm \frac{1}{4\kappa}\epsilon^{ijk}P_k(\sigma_i Q)_\alpha \tag{2.8a}$$

$$\left[L_i^{(\pm)}, \overline{Q}_{\dot\alpha}\right] = -\frac{\mathrm{i}}{2}\,\mathrm{e}^{\mp P_0/2\kappa}(\overline{Q}\sigma_i)_{\dot\alpha} \mp \frac{\mathrm{i}}{4\kappa}P_i \overline{Q}_{\dot\alpha} \pm \frac{1}{4\kappa}\epsilon^{ijk}P_k(\overline{Q}\sigma_i)_{\dot\alpha}\,. \tag{2.8b}$$

The relations (2.8a) and (2.8b) supplemented with unmodified other relations in the supercharge sector [17]

$$\begin{aligned}&\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 4\kappa\delta_{\alpha\beta}\sin\frac{P_0}{2\kappa} - 2P_i(\sigma_i)_{\alpha\dot\beta}\\ &\{Q_\alpha, Q_\beta\} = \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0\end{aligned} \tag{2.9a}$$

$$[M_i, Q_\alpha] = -\tfrac{1}{2}(\sigma_i)_\alpha^{\ \beta}Q_\beta \qquad [M_i, \overline{Q}_{\dot\alpha}] = \tfrac{1}{2}\overline{Q}_\beta(\sigma_i)^{\dot\beta}{}_{\dot\alpha} \tag{2.9b}$$

$$[P_\mu, Q_\alpha] = [P_\mu, \overline{Q}_{\dot\beta}] = 0 \tag{2.9c}$$

describe the supersymmetric extensions of the κ-Poincaré algebra $\mathcal{U}_\kappa(\mathcal{P}_4)$, given in [16].

In order to obtain the basis describing the bicrossproduct structure we introduce the following pairs of transformations:

$$\begin{aligned}N_i^{(\pm)} &= \tfrac{1}{2}\left\{L_i^{(\pm)}, \mathrm{e}^{\mp P_0/2\kappa}\right\} \mp \frac{1}{2\kappa}\epsilon_{ijk}M_j P_k\,\mathrm{e}^{\mp P_0/2\kappa}\\ P_i^{(\pm)} &= P_i\,\mathrm{e}^{\mp P_0/2\kappa}\end{aligned} \tag{2.10}$$

$$Q_\alpha^{(\pm)} = \mathrm{e}^{\mp P_0/4\kappa}Q_\alpha \qquad \overline{Q}_{\dot\alpha}^{(\pm)} = \mathrm{e}^{\pm P_0/4\kappa}\overline{Q}_{\dot\alpha}\,. \tag{2.11}$$

It should be pointed out that for the generators $(N_i^{(+)}, P_i^{(+)})$ the transformation (2.10) coincides with the one given in [19]. One obtains the following two Hopf superalgebra structures:

(i) Lorentz sector $(M_{\mu\nu}^{(\pm)} = (M_i, N_i^{(\pm)})$

(a) Algebra

$$[M_i, M_j] = \mathrm{i}\epsilon_{ijk}M_k \qquad \left[M_i, N_k^{(\pm)}\right] = \mathrm{i}\epsilon_{ijk}N_k^{(\pm)} \qquad \left[N_i^{(\pm)}, N_j^{(\pm)}\right] = -\mathrm{i}\epsilon_{ijk}M_k\,. \tag{2.12}$$

(b) Co-algebra

$$\Delta(M_i) = M_i \otimes 1 + 1 \otimes M_i \tag{2.13a}$$

$$\begin{aligned}\Delta\left(N_i^{(+)}\right) = {}& N_i^{(+)} \otimes 1 + \mathrm{e}^{-P_0/\kappa} \otimes N_i^{(+)}\\ &+\frac{1}{\kappa}\epsilon_{ijk}P_j^{(+)} \otimes M_k + \frac{\mathrm{i}}{4\kappa}(\sigma_i)_{\dot\alpha\beta}\,\mathrm{e}^{-P_0/\kappa}\overline{Q}_{\dot\alpha}^{(+)} \otimes Q_\beta^{(+)}\end{aligned} \tag{2.13b}$$

$$\begin{aligned}\Delta\left(N_i^{(-)}\right) = {}& N_i^{(-)} \otimes \mathrm{e}^{P_0/\kappa} + 1 \otimes N_i^{(-)} - \frac{1}{\kappa}\epsilon_{ijk}M_k \otimes P_j^{(-)}\\ &-\frac{i}{4\kappa}(\sigma_i)_{\dot\alpha\beta}Q_\beta^{(-)} \otimes \mathrm{e}^{P_0/\kappa}\overline{Q}_{\dot\alpha}^{(-)}\,.\end{aligned} \tag{2.13c}$$

(c) Antipode $(T_i \equiv T_i^{(+)} = T_i^{(-)})$

$$\begin{aligned}S(M_i) &= -M_i\\ S\left(N_i^{(\pm)}\right) &= -\left[N_i^{(\pm)} \pm \frac{1}{\kappa}\epsilon_{ijk}M_j P_k^{(\pm)} \mp \frac{3\mathrm{i}}{2\kappa}P_i^{(\pm)} \pm \frac{\mathrm{i}}{4\kappa}\left(2P_i^{(\pm)} - T_i\mathrm{e}^{\mp P_0/2\kappa}\right)\right]\mathrm{e}^{\pm P_0/\kappa}\,.\end{aligned} \tag{2.14}$$

(ii) 4-momentum sector: $P_\mu^{(\pm)} = (P_i^{(\pm)}, P_0^{(\pm)} = P_0)$

(a) Algebra

$$\left[P^{(\pm)}_{\mu}, P^{(\pm)}{}_{\nu}\right] = 0 \tag{2.15a}$$

$$\left[M_i, P^{(\pm)}_j\right] = \mathrm{i}\epsilon_{ijk} P^{(\pm)}_k \qquad [M_i, P_0] = 0 \tag{2.15b}$$

$$\left[N^{(\pm)}_i, P^{(\pm)}_j\right] = \pm\mathrm{i}\delta_{ij}\left[\frac{\kappa}{2}\left(1 - \mathrm{e}^{\mp 2P_0/\kappa}\right) + \frac{1}{2\kappa}\vec{P}^{(\pm)2}\right] \mp \frac{\mathrm{i}}{\kappa} P^{(\pm)}_i P^{(\pm)}_j \tag{2.15c}$$

$$\left[N^{(\pm)}_i, P_0\right] = \mathrm{i}P^{(\pm)}_i . \tag{2.15d}$$

(b) Co-algebra

$$\Delta P_0 = P_0 \otimes \mathbf{1} + \mathbf{1} \otimes P_0 \tag{2.16}$$

$$\begin{aligned} \Delta P^{(+)}_i &= P^{(+)}_i \otimes \mathbf{1} + \mathrm{e}^{-P_0/\kappa} \otimes P^{(+)}_i \\ \Delta P^{(-)}_i &= P^{(-)}_i \otimes \mathrm{e}^{P_0/\kappa} + \mathbf{1} \otimes P^{(-)}_i . \end{aligned} \tag{2.17}$$

(c) Antipode

$$S(P^{(\pm)}_i) = -\mathrm{e}^{\pm P_0/\kappa} P^{(\pm)}_i \qquad S(P_0) = -P_0 . \tag{2.18}$$

(iii) Supercharge sector $(Q^{(\pm)}_{\alpha}, \overline{Q}^{(\pm)}_{\alpha})$

(a) Algebra

$$\begin{aligned} \left[M_i, Q^{(\pm)}_{\alpha}\right] &= -\tfrac{1}{2}\left(\sigma_i Q^{(\pm)}\right)_{\alpha} \\ \left[M_i, \overline{Q}^{(\pm)}_{\dot\alpha}\right] &= \tfrac{1}{2}\left(\overline{Q}^{(\pm)}\sigma_i\right)_{\dot\alpha} \end{aligned} \tag{2.19}$$

$$\begin{aligned} \left[N^{(+)}_i, Q^{(+)}_{\alpha}\right] &= -\tfrac{1}{2}\mathrm{i}\left(\sigma_i Q^{(+)}\right)_{\alpha} \\ \left[N^{(+)}_i, \overline{Q}^{(+)}_{\dot\alpha}\right] &= -\tfrac{1}{2}\mathrm{i}\mathrm{e}^{-P_0/\kappa}\left(\overline{Q}^{(+)}\sigma_i\right)_{\dot\alpha} + \frac{1}{2\kappa}\epsilon_{ikl}P^{(+)}_k\left(\overline{Q}^{(+)}\sigma_l\right)_{\dot\alpha} \\ \left[N^{(-)}_i, Q^{(-)}_{\alpha}\right] &= -\tfrac{1}{2}\mathrm{i}\left(\sigma_i Q^{(-)}\right)_{\alpha} \\ \left[N^{(-)}_i, \overline{Q}^{(-)}_{\dot\alpha}\right] &= -\tfrac{1}{2}\mathrm{i}\mathrm{e}^{P_0/\kappa}\left(\overline{Q}^{(-)}\sigma_i\right)_{\dot\alpha} - \frac{1}{2\kappa}\epsilon_{ikl}P^{(-)}_k\left(\overline{Q}^{(-)}\sigma_l\right)_{\dot\alpha} \end{aligned} \tag{2.20}$$

$$\left[Q^{(\pm)}_{\alpha}, P^{(\pm)}_{\mu}\right] = \left[\overline{Q}^{(\pm)}_{\dot\alpha}, P^{(\pm)}_{\mu}\right] = 0 \tag{2.21}$$

$$\left\{Q^{(\pm)}_{\alpha}, \overline{Q}^{(\pm)}_{\dot\beta}\right\} = 4\kappa\delta_{\alpha\dot\beta}\sinh\left(\frac{P_0}{2\kappa}\right) - 2\,\mathrm{e}^{\pm P_0/2\kappa}P^{(\pm)}_i(\sigma_i)_{\alpha\dot\beta} . \tag{2.22}$$

(b) Co-algebra

$$\Delta(Q^{(+)}_{\alpha}) = \mathrm{e}^{-P_0/2\kappa} \otimes Q^{(+)}_{\alpha} + Q^{(+)}_{\alpha} \otimes \mathbf{1} \tag{2.23}$$

$$\Delta(\overline{Q}^{(+)}_{\dot\alpha}) = \mathbf{1} \otimes \overline{Q}^{(+)}_{\dot\alpha} + \overline{Q}^{(+)}_{\dot\alpha} \otimes \mathrm{e}^{P_0/2\kappa} \tag{2.24}$$

$$\Delta(Q^{(-)}_{\alpha}) = \mathbf{1} \otimes Q^{(-)}_{\alpha} + Q^{(-)}_{\alpha} \otimes \mathrm{e}^{P_0/2\kappa} \tag{2.25}$$

$$\Delta(\overline{Q}^{(-)}_{\dot\alpha}) = \mathrm{e}^{-P_0/2\kappa} \otimes \overline{Q}^{(-)}_{\dot\alpha} + \overline{Q}^{(-)}_{\alpha} \otimes \mathbf{1} . \tag{2.26}$$

(c) Antipodes

$$S(Q^{(\pm)}_{\alpha}) = -Q^{(\pm)}_{\alpha}\mathrm{e}^{\mp P_0/2\kappa} \qquad S(\overline{Q}^{(\pm)}_{\dot\alpha}) = -\overline{Q}^{(\pm)}_{\dot\alpha}\mathrm{e}^{\mp P_0/2\kappa} . \tag{2.27}$$

We see that we obtain two κ-deformed Poincaré Hopf superalgebras (see also (2.7); $R = 1, \ldots 14$)

$$\mathcal{U}^{(\pm)}_{\kappa}(\mathcal{P}_{4;1}) : \widetilde{I}^{(\pm)}_R = \left(M_i, N^{(\pm)}_i, P^{(\pm)}_i, P_0 Q^{(\pm)}_{\alpha}, \overline{Q}^{(\pm)}_{\dot\alpha}\right) \qquad \widetilde{I}^{(+)}_R = \left(\widetilde{I}^{(-)}_R\right)^{\oplus} \tag{2.28}$$

related by the non-standard involution $\oplus$ (see also [22]) introduced by the change of sign of κ i.e. satisfying the following properties:

$$(ab)^{\oplus} = a^{\oplus}b^{\oplus} \qquad \kappa^{\oplus} = -\kappa \qquad (a \otimes b)^{\oplus} = b^{\oplus} \otimes a^{\oplus} . \tag{2.29}$$

In the following section we shall describe the graded bicrossproduct structure of the two above written supersymmetric extensions of κ-Poincaré algebra.

3. Graded bicrossproduct structure of κ-deformed Poincaré superalgebra

Let us write the classical $N = 1$ Poincaré superalgebra as the following graded semidirect product:

$$\mathcal{P}_{4;1} = O(1,3;2) \rhd\!\!< \overline{T}_{4;2} \tag{3.1}$$

where the superalgebras $SO(1,3;2)$ and $\overline{T}_{4;2}$ are given respectively by the formulae (1.3), (1.4a) and (1.4b). The cross relations describing $O(1,3;2)$ covariance are

$$[M_{\mu\nu}, P_\rho] = \mathrm{i}(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu) \tag{3.2a}$$
$$[Q_\alpha, P_\rho] = 0 \tag{3.2b}$$
$$[M_{\mu\nu}, \overline{Q}_{\dot\alpha}] = \tfrac{1}{2}\mathrm{i}(\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\overline{Q}_{\dot\beta} \tag{3.2c}$$
$$\{Q_\alpha, \overline{Q}_{\dot\beta}\} = 2(\sigma_\mu)_{\alpha\dot\beta}P^\mu . \tag{3.2d}$$

We see that the basic superalgebra relation (3.2d) occurs as a covariance relation.

The graded semidirect product formula (3.1) can also be written as follows:

$$\mathcal{P}_{4;1} = \overline{O(1,3;2)} \rhd\!\!< T_{4;2} \tag{3.3}$$

where $\overline{O(1,3;2)} = (M_{\mu\nu}, \overline{Q}_{\dot\alpha})$ and $T_{4;2} = (P_\mu, Q_\alpha)$. It is easy to see that the relations (3.2a)–(3.2d) also describe the cross relations for (3.3), with the role of Q_α and $\overline{Q}_{\dot\alpha}$ in the bicrossproduct interchanged.

The κ-deformed Poincaré superalgebras (2.28) can be treated as κ-deformations of the semidirect products (3.1), taking the form of the bicrossproducts†. We obtain

$$\mathcal{U}_\kappa^{(+)}(\mathcal{P}_{4;1}) = T_{4;2}^{\kappa(+)} \blacktriangleright\!\!\lhd O(1,3;2) \tag{3.4a}$$
$$\mathcal{U}_\kappa^{(-)}(\mathcal{P}_{4;1}) = O(1,3;2) \rhd\!\!\blacktriangleleft T_{4;2}^{\kappa(-)} \tag{3.4b}$$

where $O(1,3;2) = (M_{\mu\nu}^{(0)}, Q_\alpha^{(0)})$ and $T_{4;2}^{\kappa(\pm)} = (P_\mu, \overline{Q}_{\dot\alpha}^{(\pm)})$. We see from the relation (2.16) and (2.23)–(2.26) that the Hopf superalgebras $T_{4;2}^{\kappa(\pm)}$ are described by the classical superalgebra and non-cocommutative coproducts; the Hopf superalgebra $O(1,3;2)$ is classical in the algebra as well as the co-algebra sectors. The bicrossproduct structure of (3.4a) and (3.4b) is described by

(i) The actions $\widehat{\alpha}^{(\pm)}$

$$\widehat{\alpha}^{(+)} \;:\; T_{4;2}^{\kappa(+)} \otimes O(1,3;2) \to T_{4;2}^{\kappa(+)} \tag{3.5a}$$
$$\widehat{\alpha}^{(-)} \;:\; O(1,3;2) \otimes T_{4;2}^{\kappa(-)} \to T_{4;2}^{\kappa(-)} \tag{3.5b}$$

modifying the cross relations (3.2a)–(3.2d).

† The bicrossproducts of Hopf algebras were introduced by Majid ([21]; see also [23]). The notion of crossproducts for braided quantum groups, which can be considered the generalization of the notion of quantum supergroups, was discussed in [24].

(ii) The coactions $\widehat{\beta}^{(\pm)}$

$$\widehat{\beta}^{(+)} : O(1,3;2) \to T_{4;2}^{\kappa(+)} \otimes O(1,3;2) \tag{3.6a}$$

$$\widehat{\beta}^{(-)} : O(1,3;2) \to O(1,3;2) \otimes T_{4;2}^{\kappa(-)} \tag{3.6b}$$

modifying the classical coproducts for the generators of $O(1,3;2)$.

Further, we shall consider only the bicrossproduct described by the action $\widehat{\alpha}^{(+)}$ and coaction $\widehat{\beta}^{(+)}$. From the formulae (2.15c) and (2.20)–(2.22) it is easy to check that $\widehat{\alpha}^{(+)}$ has the following nonlinear (i.e. κ-deformed) components†

$$\begin{aligned}
\widehat{\alpha}^{(+)}\left(P_j^{(+)} \otimes N_i^{(+)}\right) &= -\mathrm{i}\delta_{ij}\left[\frac{\kappa}{2}(1-\mathrm{e}^{-2P_0/\kappa}) + \frac{1}{2\kappa}\vec{P}^{(+)^2}\right] + \frac{\mathrm{i}}{\kappa}P_i^{(+)}P_j^{(+)} \\
\widehat{\alpha}^{(+)}\left(\overline{Q}_{\dot\alpha}^{(+)} \otimes N_i^{(+)}\right) &= \frac{\mathrm{i}}{2}\,\mathrm{e}^{-P_0/\kappa}\left(\overline{Q}^{(+)}\sigma_i\right)_{\dot\alpha} - \frac{1}{2\kappa}\epsilon_{ikl}P_k^{(+)}\left(\overline{Q}^{(+)}\sigma_i\right)_{\dot\alpha} \\
\widehat{\alpha}^{(+)}\left(\overline{Q}_{\dot\beta}^{(+)} \otimes Q_\alpha^{(+)}\right) &= 4\kappa\delta_{\alpha\dot\beta}\sinh\frac{P_0}{2\kappa} - 2\,\mathrm{e}^{P_0/2\kappa}P_i^{(+)}(\sigma_i)_{\alpha\dot\beta}\,.
\end{aligned} \tag{3.7}$$

Similarly, it follows from (2.13b) and (2.23)–(2.26) that the coaction $\widehat{\beta}^{(+)}$ has the following κ-dependent components‡:

$$\begin{aligned}
\widehat{\beta}^{(+)}\left(N_i^{(+)}\right) &= \mathrm{e}^{-P_0/\kappa} \otimes N_i^{(+)} + \frac{1}{\kappa}\mathrm{e}_{ijk}P_j^{(+)} \otimes M_k + \frac{\mathrm{i}}{4\kappa}(\sigma_i)_{\alpha\dot\beta}\mathrm{e}^{-P_0/\kappa}\overline{Q}_{\dot\alpha}^{(+)} \otimes Q_\beta^{(+)} \\
\widehat{\beta}^{(+)}\left(Q_\alpha^{(+)}\right) &= \mathrm{e}^{-P_0/2\kappa} \otimes Q_\alpha^{(+)}\,.
\end{aligned} \tag{3.8}$$

It can be checked that the actions (3.7) and coactions (3.8) satisfy the axioms required by the bicrossproducts structure [21, 23].

The action $\widehat{\alpha}^{(-)}$ and coaction $\widehat{\beta}^{(-)}$ can be obtained from $\widehat{\alpha}^{(+)}$ and $\widehat{\beta}^{(+)}$ by the non-standard involution (2.27), changing the sign of the parameter κ as well as the order in the tensor product.

Finally one can show that by modifying properly the definitions (2.10) and (2.11) of the bicrossproduct basis one can introduce the quantum $N = 1$ Poincaré superalgebra as the κ-deformed crossproduct (3.3) supplemented by suitably deformed crossproduct. In this way we obtain two other ways of expressing quantum deformation of $\mathcal{P}_{4;1}$ in the bicrossproduct form.

4. κ-deformed covariant chiral superspace

Following the general theory of bicrossproducts [21, 23, 24] from (3.4) it follows that the quantum κ-deformed $N = 1$ supergroup $(\mathcal{P}_{4;1}^{\kappa})^*$ dual to the κ-deformed Poincaré superalgebra (3.4a) and (3.4b) can also be written in the bicrossproduct form (see, for example, equation (1.6))§, where

- $(O(1,3;2))^* = C(SO(1,3;2))$ is the graded commutative algebra of functions on the graded Lorentz group $SO(1,3;2)$.
- $(T_{4;2}^{\kappa(\pm)})^*$ is the graded algebra of functions on κ-deformed chiral superspace, dual to the Hopf algebras $T_{4;2}^{\kappa(\pm)}$.

† The components which are not deformed describe standard semidirect product.

‡ In the classical case $\widehat{\beta}^{(+)}(I_A^{(0)}) = 1 \otimes I_A^{(0)}$.

§ Further, we shall use the bicrossproduct from section 3 with upper index (+), and subsequently drop this index.

The action and coaction which describe the bicrossproduct Hopf structure of the deformed graded algebra of functions on $N = 1$ Poincaré supergroup can be obtained from the actions and coactions (3.7) and (3.8) by respective dualizations.

Let us describe firstly the κ-deformed chiral superspace. Using the relations

$$\begin{aligned} \langle t, zz' \rangle &= (-1)^{\eta(z)\eta(t_{(2)})} \langle t_{(1)}, z \rangle \langle t_{(2)}, z' \rangle \\ \langle tt', z \rangle &= (-1)^{\eta(z_{(1)})\eta(t')} \langle t, z_{(1)} \rangle \langle t', z_{(2)} \rangle \end{aligned} \tag{4.1}$$

where $t, t' \in T^{\kappa}_{4;2}$ (generators P_μ, $\overline{Q}_\alpha$) and $z, z' \in \overline{T^{\kappa}_{4;2}}$ (generators x^μ, $\overline{\theta}_\alpha$) and the orthonormal basis

$$\left\langle P_\mu x^\nu \right\rangle = \delta^\nu_\mu \qquad \left\langle \overline{Q}_{\dot\alpha} \overline{\theta}^{\dot\beta} \right\rangle = \mathrm{i}\delta^{\dot\beta}_{\dot\alpha} \qquad \left\langle \overline{Q}_{\dot\alpha} x^\nu \right\rangle = \left\langle P_\mu \overline{\theta}^{\dot\beta} \right\rangle = 0 \tag{4.2}$$

one can derive that

$$\begin{aligned} \left[x^0, x^k\right] &= -\frac{1}{\kappa} x^k \qquad & \left[x^k, x^l\right] &= 0 \\ \left[x^0, \overline{\theta}^{\dot\alpha}\right] &= -\frac{1}{2\kappa} \overline{\theta}^{\dot\alpha} \qquad & \left[x^\mu, \overline{\theta}^{\dot\alpha}\right] &= \left\{\overline{\theta}^{\dot\alpha}, \overline{\theta}^{\dot\beta}\right\} = 0 \end{aligned} \tag{4.3}$$

$$\Delta x_\mu = x_\mu \otimes \mathbf{1} + \mathbf{1} \otimes x_\mu \qquad \Delta \overline{\theta}^{\dot\alpha} = \overline{\theta}^{\dot\alpha} \otimes \mathbf{1} + \mathbf{1} \otimes \overline{\theta}^{\dot\alpha}\,. \tag{4.4}$$

The relations (4.3) and (4.4) describe the chiral extension of κ-Minkowski space, introduced firstly by Zakrzewski [25]. The general formula describing the duality pairing can be written as follows (compare with [19]):

$$\langle f(P_i, P_0, \overline{\theta}_{\dot\beta}),\ :\psi(x^\mu, x^0, \overline{\theta}^\alpha):\rangle = f\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^0}, \mathrm{i}\frac{\partial}{\partial \overline{\theta}^\beta}\right) \psi\left(x^\mu, x^0, \overline{\theta}^\alpha\right)\Big|_{\substack{x^\mu = 0 \\ \overline{\theta}^{\dot\alpha} = 0}} \tag{4.5}$$

where the normal ordering describes the functions of the κ-superspace coordinates with the generators x_0 staying to the right from the generators x_i. The canonical action of $\overline{T^{\kappa}_{4;2}}$ on the κ-deformed chiral superspace is given by the well known formula

$$t \triangleright z = \langle t, z_{(1)} \rangle z_{(2)} \tag{4.6}$$

which gives $P_\mu \triangleright x^\nu = \delta_\mu{}^\nu$ and

$$\begin{aligned} P_i \triangleright :\psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right) &:=: \frac{\partial}{\partial x_i} \psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right): \\ P_0 \triangleright :\psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right) &:=: \frac{\partial}{\partial x_0} \psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right): \\ \overline{Q}_{\dot\alpha} \triangleright :\psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right) &:= i : \frac{\partial}{\partial \overline{\theta}^{\dot\alpha}} \psi\left(x^i, x^0, \overline{\theta}^{\dot\alpha}\right):\,. \end{aligned} \tag{4.7}$$

In order to describe the action of $\mathcal{U}(O(1,3;2))$ on κ-deformed superspace we dualize the action α of $\mathcal{U}(O(1,3;2))$ on Hopf superalgebra $\overline{T^{\kappa}_{4;2}}$ (see (3.7)) by the well known formula

$$\langle t, h \triangleright z \rangle = \langle \hat\alpha(h \otimes t), z \rangle\,. \tag{4.8}$$

We obtain the following covariant action of the generators of $\mathcal{U}(O(1,3;2))$ on the κ-deformed chiral superspace:

$$\begin{aligned} &M_i \triangleright x^0 = 0 \qquad && M_i \triangleright x^j = \mathrm{i}\epsilon_{ijk} x^k \qquad && M_i \triangleright \overline{\theta}^{\dot\beta} = -\tfrac{1}{2}(\sigma_i)^{\dot\beta}_{\dot\gamma} \overline{\theta}^{\dot\gamma} \\ &N_i \triangleright x^0 = -\mathrm{i}x^i \qquad && N_i \triangleright x^j = -\mathrm{i}\delta_i{}^j x^0 \qquad && N_i \triangleright \overline{\theta}^{\dot\beta} = \tfrac{1}{2}(\sigma_i)^{\dot\beta}{}_{\dot\gamma} \overline{\theta}^{\dot\gamma} \\ &Q_\alpha \triangleright x^0 = -2\mathrm{i}\overline{\theta}^{\dot\alpha} \qquad && Q_\alpha \triangleright x^j = 2\mathrm{i}(\sigma_j)_{\alpha\dot\beta} \overline{\theta}^{\dot\beta} \qquad && Q_\alpha \triangleright \overline{\theta}^{\dot\beta} = 0 \end{aligned} \tag{4.9}$$

identical with the classical κ-independent action on covariant chiral superspace [26]. Further, using the relation

$$x \triangleright (zz') = (-1)^{\eta(z)\eta(x_{(2)})}(x_{(1)} \triangleright z)(x_{(2)} \triangleright z') \tag{4.10}$$

where $x \in \mathcal{U}_\kappa(\mathcal{P}_{4;1})$, one obtains the action of the κ-deformed $N = 1$ Poincaré superalgebra generators on the functions of the κ-deformed chiral superspace coordinates. It can be checked that, for example, the action on the quadratic polynomials of $(x^k, \overline{\theta}^{\alpha})$ contains anomalous κ-dependent terms.

5. Final remarks

The aim of our considerations is to describe the quantum deformation of the supersymmetric field theory, obtained by the introduction of κ-deformed superfields formalism. It should be stressed that recently the supersymmetric models of fundamental interactions, in particular, as the 'local limits' of supersymmetric string theories, are extensively studied. The supersymmetric model which we consider as the one of particular importance is the supersymmetric extension of q-deformed gauge field theory. For the construction of such a model it is important to describe the κ-deformed graded differential calculus. Another desired property of a future κ-deformed supersymmetric formalism is its formulation in R-matrix form, in particular, for $D = 4$ κ-deformed Poincaré algebra the knowledge of its universal $\hat{R}$-matrix.

The differential calculus on the κ-deformed chiral superspace and the theory of κ-deformed superfields we plan to describe in future publications. Such a calculus is described by the supersymmetric extension of the differential calculus on κ-Minkowski space (see [27, 28]) and it is different from the one described in [29] for quantum superspace with the generators satisfying quadratic algebra.

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